Dipolar phase transitions in the cavity with Majorana populated Josephson junctions

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We found that QED in the lowest eigenstate of the microwave-cavity, capacitively coupled with Majorana populated Josephson junctions array, is described with an effective ϕ^4 model emerging in the finite domain of the system parameters. Strong anharmonicity of the cavity mode arises due to electric-dipole phase transition facilitated by the single-electron zero mode and Cooper pair tunneling between superconducting islands. An instantonic meandering of the electromagnetic field potential at finite temperatures is predicted.

I. INTRODUCTION

We start with the minimal model that describes array of topological Josephson junctions in a microwave cavity with resonant photon mode frequency ω. Majorana zero-energy modes are localized on the nanowires bridging each of the Josephson junctions consisting of two superconducting islands [1, 2]. For simplicity, we consider only odd-parity states of every separate JJ's that are coupled with each other indirectly via the microwave cavity electromagnetic field [3]. Each of the N Josephson junctions is labeled with its gauge-invariant phase difference $\gamma_i, i = 1, ..., N$, that embodies coupling to the cavity, which is described with the bare Hamiltonian $\hat{H}_{ph}$:

$$\gamma_i = \phi_i - \sqrt{2\lambda_i}\, q\,; \hat{H}_{ph} = \frac{\hbar\omega}{2}\left(p^2 + q^2\right);\ [q, p] = i\,; \tag{1}$$

Here local electric field of the (second quantized) photon mode, $\vec{E}(x)$, applied across the junction, provides coupling of the cavity to the dipole charge on the two adjacent superconducting islands, measured in the units of Cooper's pair charge $2|e|$, $\Phi_0 = hc/(2|e|)$ being the flux quantum :

$$\sqrt{2\lambda_i}\, q = \frac{2\pi}{\Phi_0}\int_i \sqrt{(hc^2)/(\omega)}\vec{E}(x)d\vec{x}\,(a+a^\dagger) \tag{2}$$

In this paper we consider single-channel Hamiltonian [1] of each pair of superconducting islands with bridging nanowires, that contain 2 localized Majorana bound states on each island with hopping term across the Josephson junction between the islands (weak link):

$$\hat{H}_{JJ} = \sum_i \begin{pmatrix} 4E_C\left(\hat{n}_i+\frac{1}{2}\right)^2 - E_J\cos\gamma_i & -E_M\,\mathrm{e}^{-\frac{i\gamma_i}{2}}\cos\frac{\gamma_i}{2} \\ -E_M\,\mathrm{e}^{\frac{i\gamma_i}{2}}\cos\frac{\gamma_i}{2} & 4E_C\left(\hat{n}_i-\frac{1}{2}\right)^2 - E_J\cos\gamma_i \end{pmatrix}, \tag{3}$$

where:

$$\hat{n}_i = -i\frac{\partial}{\partial\phi_i}\,, E_C = e^2/2C\,. \tag{4}$$

Since we consider only odd parity state of the junction with a single fermion occupying one of the islands, the shift of the cooper pair charge in Eq. (3) is $\pm 1/2$ in $2e$ units. Now, we make the canonical transformations [3], that provides interaction part of the Hamiltonian with capacitive coupling between the Josephson junctions (JJ) and cavity and dipole interaction between the JJ's in the explicit form:

$$\begin{cases} \hat{n'}_i = \hat{n}_i\,, \\ \phi'_i = \phi_i - \sqrt{2\lambda}\,q\,, \\ p' = p + \sqrt{2\lambda}\,\sum_i \hat{n}_i\,, \\ q' = q\,. \end{cases} \tag{5}$$

These canonical transformations maintain the commutation relations:

$$[\phi'_i, \hat{n}'_i] = [q', p'] = i\,, \tag{6}$$

and new p' and ϕ' are again commuting variables: $[\,p', \phi'_i\,] = 0$. For brevity, in what follows we omit $'$ in the new variables and put $\hbar = 1$. Then, after substitution of relations from Eq. (5) into Eqs. (1), (3) we find:

$$\hat{H} = \hat{H}_{ph} + \sum_i \begin{pmatrix} 4E_C\left(\hat{n}_i+\frac{1}{2}\right)^2 - gp\left(\hat{n}_i+\frac{1}{2}\right) & -E_M\,\mathrm{e}^{-\frac{i\phi_i}{2}}\cos\frac{\phi_i}{2} \\ -E_M\,\mathrm{e}^{\frac{i\phi_i}{2}}\cos\frac{\phi_i}{2} & 4E_C\left(\hat{n}_i-\frac{1}{2}\right)^2 - gp\left(\hat{n}_i-\frac{1}{2}\right) \end{pmatrix}$$

$$- \sum_i E_J\cos\phi_i\hat{1} + \frac{g^2}{2\omega}\sum_{ij}\begin{pmatrix} \left(\hat{n}_i+\frac{1}{2}\right) & 0 \\ 0 & \left(\hat{n}_i-\frac{1}{2}\right)\end{pmatrix}\begin{pmatrix} \left(\hat{n}_j+\frac{1}{2}\right) & 0 \\ 0 & \left(\hat{n}_j-\frac{1}{2}\right)\end{pmatrix} \tag{7}$$

where $g \equiv \sqrt{2\lambda}\omega$. It is remarkable, that guage-induced coupling between the JJ's and cavity is capacitive and is expressed by the second term in Eq.(7) via "induced charge" terms $\sim -gp\,(\hat{n}_i \pm 1/2)$. The dipole-dipole interaction between the JJ's is expressed by the last long-ranged coupling term with 2×2 matrices signifying fermionic states in the opposite parts of each nanowire. The Cooper pairs 2π-periodic Josephson energy is expressed by the third term in Eq.(7).

II. CHARGING ENERGY BANDS

Here we integrate out the JJ's degrees of freedom and derive effective Hamiltonian for electromagnetic field in the resonant microwave cavity that hosts the JJ's array with Majoranas described above. In this paper we restrict ourselves to the lowest JJ charge-bands splitting caused by Majorana fermions tunneling between the superconducting islands. For this purpose, we rewrite the Hamiltonian in Eq. (7) in the representation of the eigenfunctions $\exp(in\phi_i)$ of the charging energy operators $\hat{n}_i$ and restrict consideration to the case: $E_J \ll E_M \ll E_C$. It is apparent from Eq. (7) that coordinate p of the bare resonant photon cavity mode plays a role of 'induced' charge on each JJ. The charging energies then become:

$$\sim 4E_C \left(n_i \pm \frac{1}{2} - \frac{gp}{8E_C} \right)^2 - \frac{(gp)^2}{16E_C} \,. \tag{8}$$

Hence, the three bands that may overlap at the lowest energy correspond to $n_i = 0, \pm 1$, provided that simultaneously the fermionic zero mode contribution to the charge difference (called here d and expressed in the $2|e|$ units) is as follows:

$$\{n = 1, d = -1/2\}; \{n = 0, d = \pm 1/2\}; \{n = -1, d = 1/2\} \tag{9}$$

Transitions between the bands are due to terms proportional to E_M and E_J in Eq.(7). Thus, rewriting a single-JJ part of the Hamiltonian matrix in the reduced basis representation: $\exp(in\phi_i)$, $n_i = 0, \pm 1$, we find:

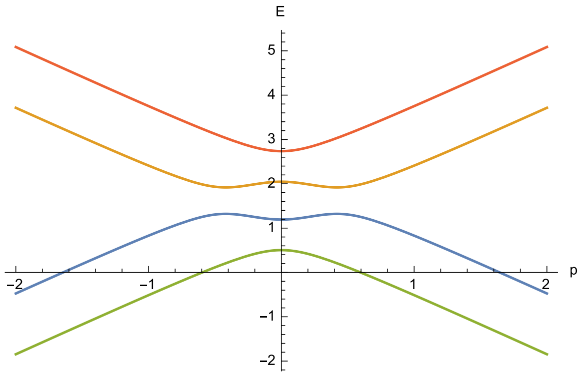

FIG. 1: The charging energy bands at $E_J = 0$ as functions of cavity induced charge $\sim gp$, see Eq.(10)

$$\hat{H}_i = \begin{pmatrix} E_C - gp/2 & -E_M/2 & 0 & E_J/2 \\ -E_M/2 & E_C - gp/2 & E_J/2 & -E_M/2 \\ 0 & E_J/2 & E_C + gp/2 & -E_M/2 \\ E_J/2 & -E_M/2 & -E_M/2 & E_C + gp/2 \end{pmatrix} \tag{10}$$

The plot of the charging energy bands (the eigenvalues of matrix Eq. (10)) is presented in Fig.1. Certainly, the induced charge periodicity of the energy bands is not seen in Fig. 1 since we aborted the number of harmonics $\exp(in\phi_i)$ by $|n| \le 1$.

III. JJ-DIPOLE ORDERING TRANSITION

The dipole-dipole interaction part of the Hamiltonian in Eq.(7) can be treated in a mean-field approximation in the zero temperature limit $T \to 0$ by a substitution of dipole operator of an i-th JJ with its expectation value in the lowest charging energy band $E_0(p)$, that possesses eigenvector $\vec{\Psi}_0(p)$ satisfying the eigenvalue equation $\hat{H}_i \vec{\Psi}_0 = E_0(p)\vec{\Psi}_0$:

$$\langle \hat{d}_i \rangle \equiv h = \left(\vec{\Psi}_0\right)^T \begin{pmatrix} \left(\hat{n}_i + \frac{1}{2}\right) & 0 \\ 0 & \left(\hat{n}_i - \frac{1}{2}\right) \end{pmatrix} \left(\vec{\Psi}_0\right)$$

Then, the dipole-dipole term ($\sim \sum_{ij}$) in Eq. (7) can be substituted with:

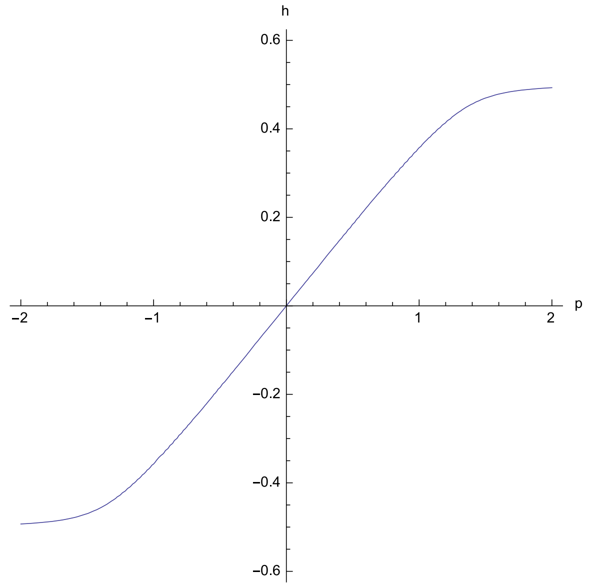

FIG. 2: The self-consistent solution $h(p)$ of Eq. (13)

$$\alpha \sum_i \begin{pmatrix} \left(\hat{n}_i + \frac{1}{2}\right) & 0 \\ 0 & \left(\hat{n}_i - \frac{1}{2}\right) \end{pmatrix} \; ; \alpha \equiv N \frac{g^2 h}{2\omega} \tag{11}$$

Finally, using above relation Eq. (11), we obtain in the mean-field approximation instead of Eq. (7) the following expression:

$$\hat{H}_{mf} = \sum_i \begin{pmatrix} 4E_C \left(\hat{n}_i + \frac{1}{2}\right)^2 - g(p-\alpha)\left(\hat{n}_i + \frac{1}{2}\right) & -E_M \, \mathrm{e}^{-\frac{i\phi_i}{2}} \cos\frac{\phi_i}{2} \\ -E_M \, \mathrm{e}^{\frac{i\phi_i}{2}} \cos\frac{\phi_i}{2} & 4E_C \left(\hat{n}_i - \frac{1}{2}\right)^2 - g(p-\alpha)\left(\hat{n}_i - \frac{1}{2}\right) \end{pmatrix} \tag{12}$$

Reducing again matrix in Eq. (12) in the lowest n_i approximation defined in Eq. (9) and solving again the eigenvalue equation $\hat{H}_i \vec{\Psi}_0(p-\alpha) = E_0(p-\alpha)\vec{\Psi}_0(p-\alpha)$, with the shift: $p \to p - \alpha$, we derive the self-consistency equation:

$$\alpha = \frac{Ng^2}{2\omega} \left(\vec{\Psi}_0(p-\alpha)\right)^T \begin{pmatrix} \left(\hat{n}_i + \frac{1}{2}\right) & 0 \\ 0 & \left(\hat{n}_i - \frac{1}{2}\right) \end{pmatrix} \left(\vec{\Psi}_0(p-\alpha)\right) \tag{13}$$

In Fig.2 one characteristic self-consistent solution $h(p) \equiv \alpha(p) 2\omega / Ng^2$ of Eq. (13) is plotted.

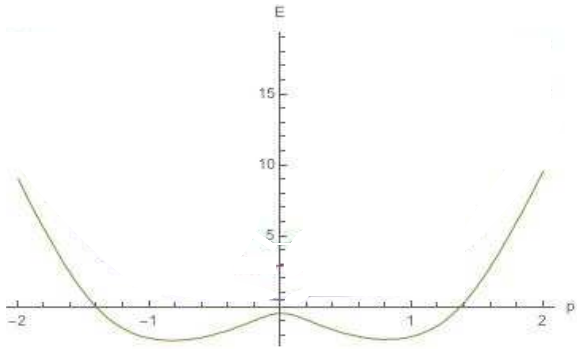

FIG. 3: The calculated effective "potential energy" $U(p)$ in Eq. (14)

IV. EFFECTIVE HAMILTONIAN OF MICROWAVE CAVITY WITH ELECTRIC DIPOLE-ORDERED JJ ARRAY

Now we integrate out the JJ's degrees of freedom of the initial Hamiltonian Eq. (7). In the law temperature limit $T \to 0$ this procedure is equivalent to a substitution of the $\hat{H}_{mf}$ in Eq. (12) with the lowest charging energy band, i.e. the smallest eigenvalue $E_0(p - h(p))$ of this matrix :

$$\hat{H}^{eff}_{ph} = \frac{\hbar\omega}{2}\left(p^2 + q^2\right) + NE_0(p - \alpha(p)) \equiv \frac{\hbar\omega}{2}q^2 + U(p) \tag{14}$$

In Fig. 3 the calculated effective "potential energy" $U(p)$ of the cavity photon is plotted. The result is remarkable, since potential is double-well and hence, there is possibility for the phase transition of the electromagnetic wave in the cavity with finite dipolar potential. Since under such transition the "left-right" symmetry is broken, the instantonic oscillations between degenerate minima of $U(p)$ [4] are possible. We shall dwell on this in detail in the forthcoming paper.

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[1] B. van Heck, F. Hassler, A. R. Akhmerov, and C. W. J. Beenakker, "Coulomb stability of the periodic Josephson effect of Majorana fermions" , Phys. Rev. B 84, 180502(R) (2011).

[2] Eran Ginossar and Eytan Grosfeld "Tunability of microwave transitions as a signature of coherent parity mixing effects in the Majorana-Transmon qubit", arXiv:1307.1159v4 , (2013).

[3] E. Almaas and D. Stroud, "Dynamics of a Josephson array in a resonant cavity", Phys. Rev. B 65, 134502 (2002).

[4] S. I. Mukhin, and M. V. Fistul, Generation of non-classical photon states in superconducting quantum metamaterials , Superconductor Science and Technology, vol. 26 , 084003 (2013).